

# NORMALITY AND $K_1$ -STABILITY OF ROY'S ELEMENTARY ORTHOGONAL GROUP

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**ABSTRACT.** In this paper, we prove the normality of the Roy's elementary orthogonal group (Dickson–Siegel–Eichler–Roy or DSER group) over a commutative ring which was introduced by A. Roy in [10] under some conditions on the hyperbolic rank. We also establish a stability theorem for  $K_1$  of Roy's group. We obtain a decomposition theorem for the elementary orthogonal group which is used to deduce the stability theorem.

## 1. INTRODUCTION

In 1960's, H. Bass initiated the study of the normal subgroup structure of linear groups. He introduced a new notion of dimension of rings, called stable rank, and proved that the principal structure theorems hold for groups whose degrees are large with respect to the stable rank. Later, J. S. Wilson, I. Z. Golubchik and A. A. Suslin made many other important contributions in this direction. In 1977, A. A. Suslin proved that over any commutative ring  $A$ , the group  $E_n(A)$  is always normal in  $GL_n(A)$  when  $n \geq 3$ .

The normal subgroup structure of symplectic and classical unitary groups over rings were studied by V. I. Kopeřko in [6], G. Taddei in [12] and by Suslin–Kopeřko in [11]. Similar results were obtained for general quadratic groups by A. Bak, V. Petrov, and G. Tang in [4], for general Hermitian groups by G. Tang in [13] and A. Bak and G. Tang in [3], and for odd unitary groups by V. Petrov in [8] and W. Yu in [15].

The stability problem for  $K_1$  of quadratic forms was studied in 1960's and in early 1970's by H. Bass, A. Bak, A. Roy, M. Kolster and L. N. Vaserstein. The stability theorems relate unitary groups and their elementary subgroups in different ranges. The stability results for quadratic  $K_1$  are due to A. Bak, V. Petrov and G. Tang (see [4]), and for Hermitian  $K_1$  are due to A. Bak and G. Tang (see [3]). Recently, in [15], W. Yu proved the  $K_1$ -stability for odd unitary groups which were introduced by V. Petrov. Stronger results for spaces over semilocal rings are due to A. Roy and M. Knebusch for quadratic spaces (see [5, 10]) and H. Reiter for Hermitian spaces (see [9]).

In this paper, we study the Dickson–Siegel–Eichler–Roy (DSER) orthogonal group over a commutative ring  $A$ , defined by A. Roy in [10]. We prove results on normality

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as well as on stability. There are three normality theorems that we will prove and the stability results will be obtained under Bass's stable range condition. A useful tool in the proof will be a decomposition theorem for the elementary subgroup that we will establish in Section 5. For proving stability, we adapt the method used in [3, 4]. We also need some commutator relations which we have proved in [1].

## 2. PRELIMINARIES

Let  $A$  be a commutative ring with identity in which 2 is invertible. Let  $Q$  be a quadratic space and  $P$  be a finitely generated projective module. Let  $M = Q \perp H(P)$ , where  $H(P)$  denote the hyperbolic space. Let  $O_A(Q \perp H(P))$  denote the orthogonal group of the quadratic space  $Q \perp H(P)$ . Here,  $Q$  is equipped with a non-singular quadratic form and  $H(P)$  has the natural hyperbolic form. Let  $EO_A(Q \perp H(P))$  be the subgroup of  $O_A(Q \perp H(P))$  generated by  $E_\alpha, E_\beta^*$  for  $\alpha \in \text{Hom}(Q, P), \beta \in \text{Hom}(Q, P^*)$ .

We now recall some basic definitions from [2]. Assume that  $Q$  and  $P$  are free modules of rank  $n$  and  $m$  respectively. In [2], by fixing bases for  $Q$  and  $P$ , we defined the maps  $\alpha_{ij} \in \text{Hom}(Q, P), \beta_{ij}^* \in \text{Hom}(Q, P^*)$  for  $\alpha \in \text{Hom}(Q, P)$  and  $\beta \in \text{Hom}(Q, P^*)$ . Then we extended these maps to  $Q \oplus P \oplus P^*$  as follows:

$$\begin{aligned}\alpha_{ij}(z, x, f) &= \eta_i \circ p_i \circ \alpha \circ \eta_j \circ p_j(z, x, f) = (0, \langle w_{ij}, z \rangle x_i, 0), \\ \beta_{ij}(z, x, f) &= \eta_i \circ p_i \circ \beta \circ \eta_j \circ p_j(z, x, f) = (0, 0, \langle v_{ij}, z \rangle f_i),\end{aligned}$$

where  $\{z_i : 1 \leq i \leq n\}$  is a basis for  $Q$ ,  $\{x_i : 1 \leq i \leq m\}$  is a basis for  $P$ ,  $\{f_i : 1 \leq i \leq m\}$  is a basis for  $P^*$  and  $w_{ij}, v_{ij} \in Q$ .

In terms of these bases, the elementary orthogonal transformations  $E_{\alpha_{ij}}$  and  $E_{\beta_{ij}}^*$  for  $1 \leq i \leq m, 1 \leq j \leq n$  are given as follows:

$$\begin{aligned}E_{\alpha_{ij}}(z, x, f) &= \left( I - \alpha_{ij}^* + \alpha_{ij} - \frac{1}{2} \alpha_{ij} \alpha_{ij}^* \right) (z, x, f) \\ &= (z - \langle f, x_i \rangle w_{ij}, x + \langle w_{ij}, z \rangle x_i - \langle f, x_i \rangle q(w_{ij}) x_i, f); \\ E_{\beta_{ij}}^*(z, x, f) &= \left( I - \beta_{ij}^* + \beta_{ij} - \frac{1}{2} \beta_{ij} \beta_{ij}^* \right) (z, x, f) \\ &= (z - \langle f_i, x \rangle v_{ij}, x, f + \langle v_{ij}, z \rangle f_i - \langle x, f_i \rangle q(v_{ij}) f_i).\end{aligned}$$

We denote  $Q \perp H(P)$  by  $Q \perp H(A)^m$ , when  $\text{rank}(P) = m$ . There is a natural embedding  $O_A(Q \perp H(A)^{m-1}) \longrightarrow O_A(Q \perp H(A)^m)$  of groups which is called the stabilization homomorphism and is given by the following matrix equation.

Let  $\begin{pmatrix} \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \mathbf{d}' & \mathbf{e}' & \mathbf{f}' \\ \mathbf{g}' & \mathbf{h}' & \mathbf{j}' \end{pmatrix}$  be an element of  $O_A(Q \perp H(A)^{(m-1)})$ . Then

$$(1) \quad \begin{pmatrix} \mathbf{a}' & \mathbf{b}' & \mathbf{c}' \\ \mathbf{d}' & \mathbf{e}' & \mathbf{f}' \\ \mathbf{g}' & \mathbf{h}' & \mathbf{j}' \end{pmatrix} \mapsto \left( \begin{array}{c|cc|cc} \mathbf{a}' & \mathbf{b}' & 0 & \mathbf{c}' & 0 \\ \hline \mathbf{d}' & \mathbf{e}' & 0 & \mathbf{f}' & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline \mathbf{g}' & \mathbf{h}' & 0 & \mathbf{j}' & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{j} \end{pmatrix}.$$

Using this equation, we define the stable orthogonal group and elementary orthogonal group as follows:

$$O_A = \lim_{m \rightarrow \infty} O_A(Q \perp H(A)^m) \text{ and}$$

$$EO_A = \lim_{m \rightarrow \infty} EO_A(Q \perp H(A)^m).$$

We now define

$$KO_{1,m}(Q \perp H(A)^m) = O_A(Q \perp H(A)^m)/EO_A(Q \perp H(A)^m),$$

which is a coset space.

We now recall some basic definitions.

**Definition 2.1** ([7, Chapter I]). Let  $A$  be a commutative ring with identity. A vector  $(a_1, \dots, a_n)$  with coefficients  $a_i \in A$  is called unimodular if there are elements  $b_1, \dots, b_n \in A$  such that

$$a_1 b_1 + \dots + a_n b_n = 1.$$

**Definition 2.2** ([14]). The ring  $A$  is said to satisfy Bass's **stable range condition**  $SA_l$  in the formulation of L. N. Vaserstein if, whenever  $(a_1, \dots, a_{l+1})$  is a unimodular vector, there exist elements  $b_1, \dots, b_l \in A$  such that  $(a_1 + a_{l+1} b_1, \dots, a_l + a_{l+1} b_l)$  is unimodular. It follows easily that  $SA_l \Rightarrow SA_k$  for any  $k \geq l$ .

**Definition 2.3** ([7, p.320]). The **stable rank**, s-rank  $A$ , of  $A$  is defined to be the smallest positive integer  $l$  such that  $A$  satisfies  $SA_l$ . If no such  $l$  exists, then the stable rank of  $A$  can be taken to be infinite. If  $A$  is a local ring, s-rank  $A = 1$ .

In this paper, we prove the following **normality theorems**.

- (i)  $O_A(Q \perp H(A)^{m-1})$  normalizes  $EO_A(Q \perp H(A)^m)$ . In particular,  $EO_A$  is a normal subgroup of  $O_A$ .
- (ii) If  $m \geq \dim \text{Max}(A) + 2$ , then  $O_A(Q \perp H(A)^m)$  normalizes  $EO_A(Q \perp H(A)^m)$ .
- (iii) If  $m > l$ , then  $O_A(Q \perp H(A)^m)$  normalizes  $EO_A(Q \perp H(A)^m)$  provided  $A$  satisfies the stable range condition  $SA_l$ .

Using these normality theorems, we establish the following stability theorem for  $K_1$ . Suppose  $A$  satisfies the stable range condition  $SA_l$ . Then, for all  $m > l$ ,  $KO_{1,m}(Q \perp H(A)^m)$  is a group. Further, the canonical map

$$KO_{1,r}(Q \perp H(A)^r) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

is surjective for  $l \leq r < m$ , and the canonical homomorphism

$$KO_{1,m}(Q \perp H(A)^m) \longrightarrow KO_{1,m+1}(Q \perp H(A)^{m+1})$$

is an isomorphism.

A key tool used in the proofs of the above theorems is a decomposition theorem for  $EO_A(Q \perp H(A)^m)$ . The decomposition involves the following subgroups.

$$\begin{aligned} C_m &= \langle [E_{\alpha_{ij}}, E_{\beta_{mk}}^*], [E_{\beta_{ij}}^*, E_{\gamma_{mk}}^*], E_{\beta_{mj}}^* : 1 \leq i < m, 1 \leq j, k \leq n \rangle, \\ D_m &= \langle [E_{\alpha_{mj}}, E_{\beta_{ik}}^*], [E_{\alpha_{mj}}, E_{\delta_{il}}], E_{\alpha_{mj}} : 1 \leq i < m, 1 \leq j, k \leq n \rangle, \\ G_m &= \langle E_{\beta_{jk}}^*, [E_{\alpha_{ir}}, E_{\beta_{jk}}^*], [E_{\beta_{ir}}^*, E_{\gamma_{jk}}^*] : 1 \leq i, j \leq m, 1 \leq r, k \leq m \rangle, \\ F_m &= \{ \eta \eta_1 : \eta \in EO_A(Q \perp H(A)^{m-1}) \text{ and } \eta_1 \in C_m \}. \end{aligned}$$

It can be easily observed that  $C_m \subseteq G_m$ .

**Definition 2.4.** Let  $\theta \in EO_A(Q \perp H(A)^m)$ , where  $Q$  has rank  $n$ . A *FDG*-decomposition of  $\theta$  is a product decomposition  $\theta = \eta \xi \mu$ , where  $\eta \in F_m$ ,  $\xi \in D_m$  and  $\mu \in G_m$ . An *FDG*-decomposition  $\theta = \eta \xi \mu$  will be called *reduced* if the  $(n+m-1, n+m)^{th}$  coefficient of  $\eta$  is 0.

In Section 5, we prove a decomposition theorem for  $EO_A(Q \perp H(A)^m)$ .

### 3. ROY'S ELEMENTARY GROUP IS NORMALIZED BY SMALLER ORTHOGONAL GROUP

In this section, we prove that  $O_A(Q \perp H(A)^{m-1})$  normalizes  $EO_A(Q \perp H(A)^m)$ .

Now, by [2, Lemma 3.4], each  $E_\alpha, E_\beta^*$  for  $\alpha \in \text{Hom}(Q, P)$  and  $\beta \in \text{Hom}(Q, P^*)$  can be written as a product of  $E_{\alpha_{ij}}, E_{\beta_{ij}}^*, 1 \leq i \leq m, 1 \leq j \leq n$ . Hence we can consider  $EO_A(Q \perp H(P))$  as the group generated by  $E_{\alpha_{ij}}$ 's and  $E_{\beta_{ij}}^*$ 's for  $\alpha \in \text{Hom}(Q, P)$  and  $\beta \in \text{Hom}(Q, P^*)$ .

Now, by [2, Lemma 4.3] and the commutator relations which we proved in [1], we note the following useful interpretation.

*The elementary orthogonal group  $EO_A(Q \perp H(P))$  is generated by the elements of the type  $E_{\alpha_{ij}}, E_{\beta_{kl}}^*, [E_{\alpha_{ij}}, E_{\delta_{kl}}], [E_{\alpha_{ij}}, E_{\beta_{kl}}^*], [E_{\gamma_{ij}}^*, E_{\beta_{kl}}^*]$  for  $1 \leq i, k \leq m, 1 \leq j, l \leq n$  and  $i \neq k$ .*

**Proposition 3.1.**  $O_A(Q \perp H(A)^{m-1})$  normalizes  $EO_A(Q \perp H(A)^m)$ .

Towards this, we recall some of the commutator relations which we proved in [1, Lemma 3.10, 3.14, 3.18, 3.19, 3.20].

**Lemma 3.2.** Let  $\alpha, \delta, \xi \in \text{Hom}(Q, P)$  and  $\beta, \gamma, \mu \in \text{Hom}(Q, P^*)$ . Then, for any given  $i, j, k, l$  such that  $1 \leq i, j, t \leq m$  and  $1 \leq k, l, r, s \leq n$ , we have the following commutator relations.

- (i)  $\left[ E_{\beta_{ik}}^*, \left[ E_{\alpha_{ir}}, E_{\gamma_{jl}}^* \right] \right] = E_{\eta_{jk}}^* \left[ E_{\nu_{jk}}^*, E_{\zeta_{ik}}^* \right]$ , where  $\eta_{jk} = -\gamma_{jl}\alpha_{ir}\beta_{ik}^*$ ,  $\nu_{jk} = -\frac{1}{2}\gamma_{jl}\alpha_{ir}\beta_{ik}^*$ ,  $\zeta_{ik} = -\beta_{ik}$  and  $i \neq j$ .
- (ii)  $\left[ E_{\beta_{ik}}^*, \left[ E_{\alpha_{ir}}, E_{\delta_{jl}} \right] \right] = E_{\lambda_{jk}} \left[ E_{\xi_{jk}}, E_{\zeta_{ik}}^* \right]$ , where  $\lambda_{jk} = \delta_{jl}\alpha_{ir}^*\beta_{ik}$ ,  $\xi_{jk} = \frac{1}{2}\delta_{jl}\alpha_{ir}^*\beta_{ik}$ ,  $\zeta_{ik} = \beta_{ik}$  and  $i \neq j$ .
- (iii)  $\left[ \left[ E_{\beta_{ir}}^*, E_{\gamma_{jl}}^* \right], \left[ E_{\alpha_{js}}, E_{\mu_{tk}}^* \right] \right] = \left[ E_{\zeta_{il}}^*, E_{\nu_{ts}}^* \right]$ , where  $\zeta_{il} = -\beta_{ir}\gamma_{jl}^*$ ,  $\nu_{ts} = \mu_{tk}\alpha_{js}^*$  and for  $i, j, t$  distinct.
- (iv)  $\left[ \left[ E_{\alpha_{ir}}, E_{\delta_{jl}} \right], \left[ E_{\xi_{tk}}, E_{\beta_{js}}^* \right] \right] = \left[ E_{\lambda_{il}}, E_{\eta_{ts}} \right]$ , where  $\lambda_{il} = \alpha_{ir}\delta_{jl}^*$ ,  $\eta_{ts} = \xi_{tk}\beta_{js}^*$  and for  $i, j, t$  distinct.
- (v)  $\left[ \left[ E_{\alpha_{ir}}, E_{\beta_{jl}}^* \right], \left[ E_{\delta_{js}}, E_{\gamma_{tk}}^* \right] \right] = \left[ E_{\eta_{il}}, E_{\mu_{ts}}^* \right]$ , where  $\eta_{il} = -\alpha_{ir}\beta_{jl}^*$ ,  $\mu_{ts} = \gamma_{tk}\delta_{js}^*$  and for  $i, j, t$  distinct.

In particular,

- (i)  $E_{\mu_{kj}}^* = \left[ E_{\beta_{mj}}^*, \left[ E_{\alpha_{mr}}, E_{\gamma_{kl}}^* \right] \right] \left[ E_{\nu_{kj}}^*, E_{\zeta_{mj}}^* \right]^{-1}$ ,
- (ii)  $E_{\lambda_{kj}} = \left[ E_{\beta_{mj}}^*, \left[ E_{\alpha_{mr}}, E_{\delta_{kl}} \right] \right] \left[ E_{\xi_{kj}}, E_{\zeta_{mj}}^* \right]^{-1}$ ,
- (iii)  $\left[ E_{\zeta_{il}}^*, E_{\nu_{ks}}^* \right] = \left[ \left[ E_{\beta_{ir}}^*, E_{\gamma_{ml}}^* \right], \left[ E_{\alpha_{ms}}, E_{\mu_{kt}}^* \right] \right]$ ,
- (iv)  $\left[ E_{\lambda_{il}}, E_{\eta_{ks}} \right] = \left[ \left[ E_{\alpha_{ir}}, E_{\delta_{ml}} \right], \left[ E_{\xi_{kt}}, E_{\beta_{js}}^* \right] \right]$ ,
- (v)  $\left[ E_{\eta_{il}}, E_{\mu_{ks}}^* \right] = \left[ \left[ E_{\alpha_{ir}}, E_{\beta_{ml}}^* \right], \left[ E_{\delta_{ms}}, E_{\gamma_{kt}}^* \right] \right]$ .

**Lemma 3.3.** The elementary orthogonal group  $\text{EO}_A(Q \perp H(A)^m)$  is generated by those elementary generators which have  $m$  as one of the subscripts.

*Proof.* The proof follows from Lemma 3.2. The relations in Lemma 3.2 show that the group  $\text{EO}_A(Q \perp H(A)^m)$  is generated by the elements of type  $E_{\alpha_{mj}}, E_{\beta_{mk}}^*, [E_{\alpha_{ij}}, E_{\beta_{mk}}^*]$ ,  $[E_{\alpha_{mj}}, E_{\beta_{ik}}^*], [E_{\alpha_{mj}}, E_{\delta_{il}}], [E_{\beta_{ij}}^*, E_{\gamma_{mk}}^*]$ , when  $Q$  and  $P$  are free  $A$ -modules.  $\square$

As a consequence of the above lemma, it follows that the groups  $D_m$  and  $C_m$  generate the elementary group  $\text{EO}_A(Q \perp H(A)^m)$ . We now prove a normality result for the elementary orthogonal group  $\text{EO}_A(Q \perp H(A)^m)$ .

**Proposition 3.4.**  $\text{O}_A(Q \perp H(A)^{m-1})$  normalizes  $\text{EO}_A(Q \perp H(A)^m)$ .

*Proof.* To prove this, it is sufficient to prove that  $D_m$  and  $C_m$  are normalized by  $\text{O}_A(Q \perp H(A)^{m-1})$ , and we do this by direct matrix calculation.

We consider the matrix representation of the elements of  $\text{O}_A(Q \perp H(A)^m)$ .

Let  $T = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} & \mathbf{j} \end{pmatrix} \in \text{O}_A(Q \perp H(A)^m)$ . Then

$$(2) \quad T^t \Psi T = \Psi,$$

where  $\Psi = \varphi \perp \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$  is the matrix of the quadratic form on  $Q \perp H(A)^m$ . Here  $\varphi$  denotes the matrix corresponding to the nondegenerate bilinear form and  $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$  is the matrix of the bilinear form on the hyperbolic space. This equation is equivalent to the following set of equations.

$$\begin{aligned} \mathbf{a}^t \varphi \mathbf{a} + \mathbf{g}^t \mathbf{d} + \mathbf{d}^t \mathbf{g} &= \varphi, & \mathbf{b}^t \varphi \mathbf{a} + \mathbf{h}^t \mathbf{d} + \mathbf{e}^t \mathbf{g} &= 0, & \mathbf{c}^t \varphi \mathbf{a} + \mathbf{j}^t \mathbf{d} + \mathbf{f}^t \mathbf{g} &= 0, \\ \mathbf{a}^t \varphi \mathbf{b} + \mathbf{g}^t \mathbf{e} + \mathbf{d}^t \mathbf{h} &= 0, & \mathbf{b}^t \varphi \mathbf{b} + \mathbf{h}^t \mathbf{e} + \mathbf{e}^t \mathbf{h} &= 0, & \mathbf{c}^t \varphi \mathbf{b} + \mathbf{j}^t \mathbf{e} + \mathbf{f}^t \mathbf{h} &= I, \\ \mathbf{a}^t \varphi \mathbf{c} + \mathbf{g}^t \mathbf{f} + \mathbf{d}^t \mathbf{j} &= 0, & \mathbf{b}^t \varphi \mathbf{c} + \mathbf{h}^t \mathbf{f} + \mathbf{e}^t \mathbf{j} &= I, & \mathbf{c}^t \varphi \mathbf{c} + \mathbf{j}^t \mathbf{f} + \mathbf{f}^t \mathbf{j} &= 0. \end{aligned}$$

These equations are equivalent to the equation

$$T^{-1} = \begin{pmatrix} \varphi^{-1} \mathbf{a}^t \varphi & \varphi^{-1} \mathbf{g}^t & \varphi^{-1} \mathbf{d}^t \\ \mathbf{c}^t \varphi & \mathbf{j}^t & \mathbf{f}^t \\ \mathbf{b}^t \varphi & \mathbf{h}^t & \mathbf{e}^t \end{pmatrix}.$$

We now consider the generators for the subgroups  $D_m$  and  $C_m$  of  $\text{EO}_A(Q \perp H(A)^m)$  and prove that they are normalized by an element in  $\text{O}_A(Q \perp H(A)^{m-1})$ .

Consider  $T \in \text{O}_A(Q \perp H(A)^{m-1})$  as an element in  $\text{O}_A(Q \perp H(A)^m)$  by the stabilization homomorphism. Then we conjugate the elementary generators of  $\text{EO}_A(Q \perp H(A)^m)$  and write the conjugated element as a product of elementary generators.

Corresponding to the elementary generator  $E_{\alpha_{mj}}$ , we have

$$\begin{aligned} T^{-1} E_{\alpha_{mj}} T &= \begin{pmatrix} I & 0 & -\phi^{-1} \mathbf{a}^t \alpha_{mj}^t \mathbf{j} \\ \mathbf{j}^t \alpha_{mj} \mathbf{a} & I + \mathbf{j}^t \alpha_{mj} \mathbf{b} & \mathbf{j}^t \alpha_{mj} \mathbf{c} - \mathbf{c}^t \alpha_{mj}^t \mathbf{j} - \frac{1}{2} \mathbf{j}^t \alpha_{mj} \alpha_{mj}^* \mathbf{j} \\ 0 & 0 & I - \mathbf{b}^t \alpha_{mj}^t \mathbf{j} \end{pmatrix} \\ &= [E_{\mathbf{j}^t \alpha_{mj} \mathbf{b} \mathbf{c}^t \phi}, E_{\frac{\mathbf{j}^t \alpha_{mj}}{2}}] [E_{\mathbf{c}^t \phi}, E_{\mathbf{j}^t \alpha_{mj}}] [E_{\mathbf{b}^t \phi}^*, E_{\mathbf{j}^t \alpha_{mj}}] E_{\mathbf{j}^t \alpha_{mj} \mathbf{a}}. \end{aligned}$$

Corresponding to the elementary generator  $E_{\beta_{mj}}^*$ , we have

$$\begin{aligned} T^{-1} E_{\beta_{mj}}^* T &= \begin{pmatrix} I & -\phi^{-1} \mathbf{a}^t \beta_{mj}^t \mathbf{e} & 0 \\ 0 & I - \mathbf{c}^t \beta_{mj}^t \mathbf{e} & 0 \\ \mathbf{e}^t \beta_{mj} \mathbf{a} & \mathbf{e}^t \beta_{mj} \mathbf{b} - \mathbf{b}^t \beta_{mj}^t \mathbf{e} - \frac{1}{2} \mathbf{e}^t \beta_{mj} \beta_{mj}^* \mathbf{e} & I + \mathbf{e}^t \beta_{mj} \mathbf{c} \end{pmatrix} \\ &= \left[ E_{\mathbf{e}^t \beta_{mj} \mathbf{c} \mathbf{b}^t \phi}^*, E_{\frac{\mathbf{e}^t \beta_{mj}}{2}}^* \right] \left[ E_{\mathbf{b}^t \phi}^*, E_{\mathbf{e}^t \beta_{mj}}^* \right] \left[ E_{\mathbf{c}^t \phi}, E_{\mathbf{e}^t \beta_{mj}}^* \right] E_{(\mathbf{e}^t \beta_{mj} \mathbf{a})}. \end{aligned}$$

Corresponding to the elementary generator  $[E_{\alpha_{mj}}, E_{\beta_{kl}}^*]$ , we have

$$\begin{aligned}
T^{-1}[E_{\alpha_{mj}}, E_{\beta_{kl}}^*]T &= \begin{pmatrix} I & 0 & \phi^{-1}(\mathbf{d}^t \beta_{kl} \alpha_{mj}^* \mathbf{j})^t \\ -\mathbf{j}^t \alpha_{mj} \phi^{-1} \beta_{kl}^t \mathbf{d} & I - \mathbf{j}^t \alpha_{mj} \beta_{kl}^* \mathbf{e} & \mathbf{f}^t \beta_{kl} \phi^{-1} \alpha_{mj}^t \mathbf{j} - \mathbf{j}^t \alpha_{mj} \phi^{-1} \beta_{kl}^t \mathbf{f} \\ 0 & 0 & I + \mathbf{e}^t \beta_{kl} \alpha_{mj}^* \mathbf{j} \end{pmatrix} \\
&= \left[ E_{(\frac{\mathbf{j}^t \alpha_{mj}}{2})}, E_{(\mathbf{j}^t \alpha_{mj} \phi^{-1} \beta_{kl}^t \mathbf{f} \mathbf{e}^t \beta_{kl})} \right] \left[ E_{(\mathbf{j}^t \alpha_{mj})}, E_{(\mathbf{f}^t \beta_{kl})} \right] \\
&\quad \left[ E_{(\mathbf{j}^t \alpha_{mj})}, E_{(\mathbf{e}^t \beta_{kl})}^* \right] E_{-(\mathbf{j}^t \alpha_{mj} \phi^{-1} \beta_{kl}^t \mathbf{d})}.
\end{aligned}$$

Corresponding to the elementary generator  $[E_{\alpha_{ij}}, E_{\beta_{mk}}^*]$ , we have

$$\begin{aligned}
T^{-1}[E_{\alpha_{ij}}, E_{\beta_{mk}}^*]T &= \begin{pmatrix} I & -\phi^{-1}(\mathbf{e}^t \beta_{mk} \alpha_{ij}^* \mathbf{g})^t & 0 \\ 0 & I - \mathbf{j}^t \alpha_{ij} \beta_{mk}^* \mathbf{e} & 0 \\ \mathbf{e}^t \beta_{mk} \alpha_{ij}^* \mathbf{g} & \mathbf{e}^t \beta_{mk} \alpha_{ij}^* \mathbf{h} - \mathbf{h}^t \alpha_{ij} \beta_{mk}^* \mathbf{e} & I + \mathbf{e}^t \beta_{mk} \alpha_{ij}^* \mathbf{j} \end{pmatrix} \\
&= \left[ E_{(\mathbf{e}^t \beta_{mk} \phi^{-1} \alpha_{ij}^t \mathbf{j} \mathbf{h}^t \alpha_{ij})}, E_{(\frac{\mathbf{e}^t \beta_{mk}}{2})}^* \right] \left[ E_{(\mathbf{h}^t \alpha_{ij})}, E_{(\mathbf{e}^t \beta_{mk})}^* \right] \\
&\quad \left[ E_{(\mathbf{j}^t \alpha_{ij})}, E_{(\mathbf{e}^t \beta_{mk})}^* \right] E_{(\mathbf{e}^t \beta_{mk} \phi^{-1} \alpha_{ij}^t \mathbf{g})}.
\end{aligned}$$

Corresponding to the elementary generator  $[E_{\alpha_{mk}}, E_{\delta_{jl}}]$ , we have

$$\begin{aligned}
T^{-1}[E_{\alpha_{mk}}, E_{\delta_{jl}}]T &= \begin{pmatrix} I & 0 & \phi^{-1} \mathbf{g}^t \delta_{jl} \alpha_{mk}^* \mathbf{j} \\ -\mathbf{j}^t \alpha_{mk} \delta_{jl}^* \mathbf{g} & I - \mathbf{j}^t \alpha_{mk} \delta_{jl}^* \mathbf{h} & \mathbf{j}^t (\delta_{jl} \alpha_{mk}^* - \alpha_{mk} \delta_{jl}^*) \mathbf{j} \\ 0 & 0 & I + \mathbf{h}^t \delta_{jl} \alpha_{mk}^* \mathbf{j} \end{pmatrix} \\
&= \left[ E_{(\frac{1}{2} \mathbf{j}^t \alpha_{mk})}, E_{(\mathbf{j}^t \alpha_{mk} \delta_{jl}^* \mathbf{h} \mathbf{j}^t \delta_{jl})} \right] \left[ E_{(\mathbf{j}^t \alpha_{mk})}, E_{(\mathbf{h}^t \delta_{jl})}^* \right] \left[ E_{(\alpha_{mk})}, E_{(\mathbf{j}^t \delta_{jl})} \right] E_{(-\mathbf{j}^t \alpha_{mk} \delta_{jl}^* \mathbf{g})}.
\end{aligned}$$

Corresponding to the elementary generator  $[E_{\beta_{mk}}^*, E_{\gamma_{jl}}]$ , we have

$$\begin{aligned}
T^{-1}[E_{\beta_{mk}}^*, E_{\gamma_{jl}}]T &= \begin{pmatrix} I & \phi^{-1} \mathbf{d}^t \gamma_{jl} \phi^{-1} \beta_{mk}^t \mathbf{e} & 0 \\ 0 & I + \mathbf{f}^t \gamma_{jl} \phi^{-1} \beta_{mk}^t \mathbf{e} & 0 \\ -\mathbf{e}^t \beta_{mk} \phi^{-1} \gamma_{jl}^t \mathbf{d} & \mathbf{e}^t (\gamma_{jl} \beta_{mk}^* - \beta_{mk} \gamma_{jl}^*) \mathbf{e} & I - \mathbf{e}^t \beta_{mj} \phi^{-1} \gamma_{jl}^t \mathbf{f} \end{pmatrix} \\
&= [E_{(\frac{\mathbf{e}^t \beta_{mk}}{2})}^*, E_{(\mathbf{e}^t \beta_{mk} \gamma_{jl}^* \mathbf{e} \mathbf{f}^t \gamma_{jl})}^*] [E_{(\mathbf{e}^t \beta_{mk})}^*, E_{(\mathbf{e}^t \gamma_{jl})}^*] \\
&\quad [E_{(\mathbf{e}^t \beta_{mk})}^*, E_{(\mathbf{f}^t \gamma_{jl})}^*] E_{(-\mathbf{e}^t \beta_{mk} \gamma_{jl}^* \mathbf{d})}.
\end{aligned}$$

Now it follows from the above equations that  $C_m$  and  $D_m$  are normalized by  $O_A(Q \perp H(A)^{m-1})$ . Hence the proposition follows.  $\square$

**Corollary 3.5.**  $EO_A$  is a normal subgroup of  $O_A$ .

#### 4. NORMALITY OF ROY'S ELEMENTARY GROUP UNDER CONDITION ON HYPERBOLIC RANK

In this section, we prove that  $\text{EO}_A(Q \perp H(A)^m)$  is normal in  $\text{O}_A(Q \perp H(A)^m)$  under a condition on the hyperbolic rank. First, we prove the normality when the hyperbolic rank is at least  $d + 2$ , where  $d = \dim \text{Max}(A)$ .

**Theorem 4.1.**  *$\text{EO}_A(Q \perp H(A)^m)$  is normal in  $\text{O}_A(Q \perp H(A)^m)$  when  $m \geq d + 2$ , where  $d = \dim \text{Max}(A)$ .*

*Proof.* By [10, Theorem 7.1], it follows that  $\text{EO}_A(Q \perp H(A)^m)$  acts transitively on hyperbolic pairs. In the case of semi-local rings, by [10, Theorem 8.1'], the same holds for  $m \geq 1$ .

For, if  $\alpha \in \text{O}_A(Q \perp H(A)^m)$  and  $(e_1, f_1)$  is a hyperbolic pair, then, by [10, Corollary 6.4],  $(\alpha e_1, \alpha f_1)$  and  $(e_1, f_1)$  are in the same orbit of  $\text{EO}_A(Q \perp H(A)^m)$ . Let  $e$  be a map which takes one orbit to the other. Therefore  $e\alpha$  fixes  $(e_1, f_1)$  and hence  $e\alpha \in \text{O}_A(Q \perp H(A)^{m-1})$ , whence so does  $(e\alpha)^{-1}$ . Now, by Proposition 3.4, it follows that  $(e\alpha)^{-1}$  normalizes the elementary orthogonal group  $\text{EO}_A(Q \perp H(A)^m)$ . This implies that  $\alpha^{-1}$  normalizes  $\text{EO}_A(Q \perp H(A)^m)$ .  $\square$

#### 5. A DECOMPOSITION THEOREM

In this section, we prove a decomposition of Roy's elementary orthogonal group under the stable range condition. We start with the following lemma.

**Lemma 5.1.** *Let  $m \geq l + 1$ . Then, for any  $\sigma \in \text{O}_A(Q \perp H(A)^m)$ , there is an element  $\varrho \in G_m$  such that  $\sigma\varrho$  has 1 in its  $(n + m, n + m)^{\text{th}}$  position.*

*Proof.* Let  $\sigma$  be the  $3 \times 3$  block matrix corresponding to the orthogonal transformation  $\sigma \in \text{O}_A(Q \perp H(A)^m)$  given by

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

where  $\sigma_{11}$  is an  $n \times n$  matrix,  $\sigma_{12}, \sigma_{13}$  are  $n \times m$  matrices,  $\sigma_{21}, \sigma_{31}$  are  $m \times n$  matrices and  $\sigma_{22}, \sigma_{23}, \sigma_{32}, \sigma_{33}$  are  $m \times m$  matrices. Since  $\sigma^{-1} \in \text{O}_A(Q \perp H(A)^m)$ , it also has a similar matrix description. Now  $(\sigma_{21}, \sigma_{22}, \sigma_{23})$  is a unimodular vector in  $M_n(A) \times (M_m(A))^2$ . Let  $(u, v, w)$  be the bottom row of  $(\sigma_{21}, \sigma_{22}, \sigma_{23})$ . Since  $\sigma^{-1} \in \text{O}_A(Q \perp H(A)^m)$ , it also has a similar matrix description. Now  $(\sigma_{21}, \sigma_{22}, \sigma_{23})$  is a unimodular vector in  $M_n(A) \times (M_m(A))^2$ . Let  $(u, v, w)$  be the bottom row of  $(\sigma_{21}, \sigma_{22}, \sigma_{23})$ . It is unimodular in  $A^n \times A^{2m}$ . Also,  $v$  is unimodular in  $A^m \cong P$ . Then, by [10, Remark 5.6], there exists an orthogonal transformation  $\mu_1 = E_\beta^* \in G_m$  which maps  $(u, v, w)$  into a unimodular vector  $(0, v, w') \in H(P)$ .



Since  $A$  satisfies the stable range condition  $SA_l$  and  $m \geq l + 1$ , there exists a matrix  $\gamma \in M_m(A)$  such that  $v' + w'\gamma$  is unimodular in  $A^m$ . Now set

$$\mu_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \gamma & I \end{pmatrix} \in G_m,$$

where  $I$  denotes the identity matrix and  $0$  denotes the zero matrix of the corresponding block size.

Since  $A$  satisfies stable range condition  $SA_l$  and  $m \geq l + 1$ , there is a product  $\epsilon$  of elementary matrices such that  $(v' + w'\gamma)\epsilon = (0, \dots, 0, 1)$ .

Set

$$\mu_3 = \begin{pmatrix} I & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^{t-1} \end{pmatrix} \in G_m.$$

Then  $\sigma\mu_1\mu_2\mu_3$  has  $(n + m)^{th}$  row  $(0, 0, \dots, 1, w\epsilon^{t-1})$ . This completes the proof of the lemma.  $\square$

Now we can prove the following decomposition theorem.

**Theorem 5.2 (Decomposition Theorem).** *Let  $m \geq l + 2$ . Then every element of  $\text{EO}_A(Q \perp H(A)^m)$  has a reduced FDG-decomposition.*

*Proof.* We first show that if  $\theta$  has an FDG-decomposition, then it has a reduced one.

Let  $\eta\xi\mu$  be an FDG-decomposition of  $\theta$ . Write

$$\eta = \begin{pmatrix} \eta_{11} & \eta_{12} & 0 & \eta_{14} & 0 \\ \eta_{21} & \eta_{22} & 0 & \eta_{24} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \eta_{41} & \eta_{42} & 0 & \eta_{44} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the block matrix  $\eta_{11}$  is of size  $n \times n$ , the block matrices  $\eta_{12}, \eta_{14}$  are of size  $n \times (m - 1)$ ,  $\eta_{21}, \eta_{41}$  are of size  $(m - 1) \times n$  and  $\eta_{22}, \eta_{24}, \eta_{42}, \eta_{44}$  are of size  $(m - 1) \times (m - 1)$ . and set

$$\eta_1 = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{14} \\ \eta_{21} & \eta_{22} & \eta_{24} \\ \eta_{41} & \eta_{42} & \eta_{44} \end{pmatrix}.$$

By definition,  $\eta_1 \in \text{EO}_A(Q \perp H(A)^{m-1})$ . Since  $m \geq l + 2$ , it follows from Lemma 5.1 that there is an element  $\mu_1 \in G_{m-1}$  such that the  $(n + m - 1, n + m - 1)^{th}$  coefficient of  $\eta_1\mu_1$  is 1. Identifying  $\mu_1$  with its image under the stabilization map

$$\text{EO}_A(Q \perp H(A)^{m-1}) \longrightarrow \text{EO}_A(Q \perp H(A)^{m-1} \perp H(A)),$$

we have  $\mu_1 \in G_m$  and the  $(n + m - 1, n + m - 1)^{th}$  coefficient of  $\eta\mu_1$  is 1. Also,  $\mu_1 \in F_m \cap G_m$  and  $\mu_1$  normalizes  $D_m$ . Thus  $(\eta\mu_1)(\mu_1^{-1}\xi\mu_1)(\mu_1^{-1}\mu)$  is an FDG-decomposition of  $\theta$  such that the  $(n + m - 1, n + m - 1)^{th}$  coefficient of  $\eta\mu_1$  is 1.

Choose an element  $\alpha \in \text{Hom}_A(Q, P)$  such that the  $(n+m-1, n+m)^{\text{th}}$  coefficient of  $\eta\mu_1[E_{\alpha_{m-1,j}}, E_{\beta_{mk}}^*]$  is 0. Choose  $\delta \in \text{Hom}_A(Q, P)$  such that the  $(n+m, n+m-1)^{\text{th}}$  coefficient of  $\mu_1^{-1}\xi\mu_1[E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]$  is 0. Let  $\mu_2 = [E_{\alpha_{m-1,j}}, E_{\beta_{mk}}^*]$  and  $\mu_3 = [E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]$ . Then

$$\mu_2^{-1}(\mu_1^{-1}\xi\mu_1\mu_3)\mu_2 = \eta_2\xi_1$$

for some  $\eta_2 \in \text{EO}_A(Q \perp H(A)^{m-1}) \subseteq F_m$  and some  $\xi \in D_m$ . Thus

$$\theta = \eta\xi\mu = (\eta\mu_1\mu_2)(\mu_2^{-1}(\mu_1^{-1}\xi\mu_1\mu_3)\mu_2)(\mu_2^{-1}\mu_3^{-1}\mu_1^{-1}\mu)$$

which is a reduced  $FDG$ -decomposition of  $\theta$ .

We now prove that every element of  $\text{EO}_A(Q \perp H(A)^m)$  does have an  $FDG$ -decomposition. In order to do this, we consider the generators  $\varepsilon$  of  $\text{EO}_A(Q \perp H(A)^m)$  and show that  $\varepsilon F_m D_m G_m \subseteq F_m D_m G_m$ . It would follow from this that  $\text{EO}_A(Q \perp H(A)^m) = F_m D_m G_m$ . Thus, by the first part of the proof, it is enough to prove that  $\varepsilon\eta\xi\mu \in F_m D_m G_m$  for each reduced  $FDG$ -decomposition  $\eta\xi\mu$ . The rest of the proof shows this.

The commutator relations in the Lemma 3.2 show that  $F_m$  and the matrices  $[E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]$  generate  $\text{EO}_A(Q \perp H(A)^m)$ , where  $\text{rank}(Q) = n$ . Evidently,

$$F_m(F_m D_m G_m) \subseteq F_m D_m G_m.$$

We now consider an element with a reduced  $FDG$ -decomposition  $\eta\xi\mu$ . Since the  $(n+m-1, n+m)^{\text{th}}$  coefficient of  $\eta$  is 0,  $\eta$  can be expressed as a product  $\eta = \eta_3\eta_4$ , where  $\eta_3 \in C_m$  such that the  $(n+m-1, n+m)^{\text{th}}$  coefficient of  $\eta_3$  is 0 and  $\eta_4 \in \text{EO}_A(Q \perp H(A)^m)$ . By a straightforward computation, one can show that

$$[E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]\eta_3[E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]^{-1} \in F_m.$$

Clearly,  $\text{EO}_A(Q \perp H(A)^m)$  normalizes  $D_m$ . Thus

$$[E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]\eta\xi\mu = ([E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]\eta_3[E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]^{-1}\eta_4)(\eta_4^{-1}[E_{\delta_{mj}}, E_{\beta_{m-1,k}}^*]\eta_4\xi)\mu$$

which is an  $FDG$ -decomposition.  $\square$

## 6. NORMALITY UNDER STABLE RANGE

In this section, we prove the normality under the assumption that  $A$  satisfies the stable range condition  $SA_l$ .

**Theorem 6.1.** *Let  $A$  be a commutative ring in which 2 is invertible. Suppose  $A$  satisfies the stable range condition  $SA_l$ . Then, for all  $m > l$ ,  $\text{EO}_A(Q \perp H(A)^m)$  is normal in  $\text{O}_A(Q \perp H(A)^m)$ .*

*Proof.* Let  $\eta \in \text{EO}_A(Q \perp H(A)^m)$ , where  $\text{rank}(Q) = n$ . By Lemma 5.1, there is an element  $\varrho_1$  in  $G_m \subseteq \text{EO}_A(Q \perp H(A)^m)$  such that the  $(n+m, n+m)^{\text{th}}$  coefficient of  $\eta\varrho_1$  is 1. Then there is a matrix  $\varrho_2 = \prod_{i=1}^{m-1}[E_{\alpha_{mj}}, E_{\beta_{ik}}^*]$  such that  $\eta\varrho_1\varrho_2$  has 0 in the first  $n+m-1$  entries of its  $(n+m)^{\text{th}}$  row and 1 in the  $(n+m)^{\text{th}}$  entry of this row. It follows that there is a matrix  $\varrho_3 = \prod_{i=1}^m[E_{\beta_{ir}}^*, E_{\gamma_{mk}}^*] \prod_{i=1}^{m-1}[E_{\alpha_{ir}}, E_{\beta_{mk}}^*]E_{\gamma_{mj}}^*$  such that  $\varrho_3\eta\varrho_1\varrho_2$

has the same  $m^{th}$  row as  $\eta\varrho_1\varrho_2$  and the same  $m^{th}$  column as the  $(n+2m) \times (n+2m)$  identity matrix. For any matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \in O_A(Q \perp H(A)^m),$$

it follows from the Equation (2) that we get the  $(n+2m, n+2m)^{th}$  coefficient of  $\varrho_3\eta\varrho_1\varrho_2$  is 1. Then there is a matrix

$$\varrho_4 = \prod_{i=1}^{m-1} [E_{\alpha_{mk}}, E_{\beta_{ir}}^*] \prod_{i=1}^m [E_{\beta_{ir}}^*, E_{\gamma_{mk}}^*] \prod_{i=1}^m [E_{\alpha_{ir}}, E_{\delta_{mk}}] E_{\zeta_{mj}}$$

such that  $\varrho_4\varrho_3\eta\varrho_1\varrho_2$  has the same  $(n+m)^{th}$  row and  $(n+m)^{th}$  column as  $\varrho_3\eta\varrho_1\varrho_2$  and the same  $(n+2m)^{th}$  column as the  $(n+2m, n+2m)$  identity matrix. Now, it follows that  $\varrho_4\varrho_3\eta\varrho_1\varrho_2$  has the same  $(n+2m)^{th}$  row as the  $(n+2m, n+2m)$  identity matrix. Thus, by the stabilization homomorphism, we have  $\varrho_4\varrho_3\eta\varrho_1\varrho_2 \in O_A(Q \perp H(A)^{m-1})$ , where  $\text{rank}(Q) = n$ . Let  $\rho = \varrho_4\varrho_3\eta\varrho_1\varrho_2$ . By Proposition 3.4, it follows that  $\rho$  normalizes  $EO_A(Q \perp H(A)^m)$ , where  $\text{rank}(Q) = n$ . Since  $\eta = \varrho_3^{-1}\varrho_4^{-1}\rho\varrho_2^{-1}\varrho_1^{-1}$ , it follows that  $\eta$  normalizes  $EO_A(Q \perp H(A)^m)$ . Thus  $EO_A(Q \perp H(A)^m)$  is normal in  $O_A(Q \perp H(A)^m)$ .  $\square$

## 7. STABILITY OF $K_1$

In this section, we prove the following stability theorem using the normality theorem of the previous section and the decomposition theorem.

**Theorem 7.1.** *Let  $A$  be a commutative ring of stable rank  $l$  in which 2 is invertible. Then, for all  $m > l$ ,  $KO_{1,m}(Q \perp H(A)^m)$  is a group. Further, the canonical map*

$$KO_{1,r}(Q \perp H(A)^r) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

*is surjective for  $l \leq r < m$ , and the canonical homomorphism*

$$KO_{1,m}(Q \perp H(A)^m) \longrightarrow KO_{1,m+1}(Q \perp H(A)^{m+1})$$

*is an isomorphism.*

*Proof.* By Theorem 6.1, we get  $KO_{1,m}(Q \perp H(A)^m)$  is a group and the map

$$KO_{1,m-1}(Q \perp H(A)^{m-1}) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

is surjective. By induction on  $m-l$ , we obtain that the map

$$KO_{1,r}(Q \perp H(A)^r) \longrightarrow KO_{1,m}(Q \perp H(A)^m)$$

is surjective for  $l \leq r < m$ .

To prove the final assertion, let  $\sigma \in O_A(Q \perp H(A)^m) \cap EO_A(Q \perp H(A)^m \perp H(A))$ . Let  $\eta\xi\mu$  be an  $F_{(m+1)}D_{(m+1)}G_{(m+1)}$ -decomposition of  $\sigma$ . Since the  $(n+m+1)^{th}$  row of  $\eta$  coincides with that of the  $(n+2(m+1)) \times (n+2(m+1))$  identity matrix, it follows

that the  $(n + m + 1)^{th}$  row of  $\eta\xi\mu$  coincides with the  $(n + m + 1)^{th}$  row of  $\xi\mu$ . Thus the  $(n + m + 1)^{th}$  row of  $\xi\mu$  coincides with that of the  $(n + 2(m + 1)) \times (n + 2(m + 1))$  identity matrix. We can write the matrix  $\mu$  as

$$\mu = \begin{pmatrix} I & \gamma & 0 \\ 0 & \varepsilon & 0 \\ \vartheta & \psi & \varepsilon^{t^{-1}} \end{pmatrix},$$

where  $I$  is an  $n \times n$  identity matrix,  $\gamma$  is an  $n \times m$  matrix,  $\varepsilon$  is an  $m \times m$  invertible matrix,  $\vartheta$  and  $\psi$  are matrices of size  $m \times n$  and  $m \times m$  respectively.

If  $(u, v, w)$  denotes the  $(n + m + 1)^{th}$  row of  $\xi$ , then the  $(n + m + 1)^{th}$  row of  $\xi\mu$  is

$$(u, \quad v, \quad w) \begin{pmatrix} I & \gamma & 0 \\ 0 & \varepsilon & 0 \\ \vartheta & \psi & \varepsilon^{t^{-1}} \end{pmatrix} = (u + w\vartheta, \quad u\gamma + v\varepsilon + w\psi, \quad w\varepsilon^{t^{-1}})$$

The  $(n + m + 1)^{th}$  row of  $\xi\mu$  coincides with that of the  $n + 2(m + 1) \times n + 2(m + 1)$  identity matrix. Hence  $w(\varepsilon^t)^{-1} = 0$ . Since  $(\varepsilon^t)^{-1}$  is invertible, we get  $w = 0$ . This implies that  $u = 0$ . Thus  $\xi \in G_{m+1}$ .

Now write  $\eta = \eta_1\mu_1$ , where  $\eta_1 \in \text{EO}_A(Q \perp H(A)^m)$  and  $\mu_1 \in C_{m+1} \subseteq G_{m+1}$ .

Then  $\sigma = \eta_1\mu_1\xi\mu$  and  $\mu_1\xi\mu \in G_{m+1} \cap \text{O}_A(Q \perp H(A)^m)$ . It suffices to show that  $\mu_1\xi\mu \in \text{EO}_A(Q \perp H(A)^m)$ . In fact, we show that  $\mu_1\xi\mu \in G_m$ .

Write

$$\mu_1\xi\mu = \begin{pmatrix} I & \gamma & 0 \\ 0 & \varepsilon & 0 \\ \vartheta & \delta & \varepsilon^{t^{-1}} \end{pmatrix}.$$

Since  $\mu_1\xi\mu \in \text{O}_A(Q \perp H(A)^m)$ , it follows that  $\gamma, \delta$  have their last column 0 and  $\vartheta, \delta$  have their last row 0. Also, it follows that  $\varepsilon \in \text{GL}_m(A)$ . From the definition of  $G_{m+1}$ , we see that  $\varepsilon$  is an  $(m + 1) \times (m + 1)$  matrix of the form

$$\varepsilon = \begin{pmatrix} \varepsilon' & 0 \\ 0 & 1 \end{pmatrix} \in \text{E}_{m+1}(A)$$

Thus  $\varepsilon' \in \text{E}_{m+1}(A) \cap \text{GL}_m(A)$ . Since  $A$  satisfies the stable range condition, by the stability for  $K_1$  of the general linear group, we have  $\varepsilon' \in \text{E}_m(A)$ .

Thus  $\mu_1\xi\mu$  lies in  $G_m$ . Hence the canonical homomorphism

$$KO_{1,m}(Q \perp H(A)^m) \longrightarrow KO_{1,m+1}(Q \perp H(A)^m)$$

is an isomorphism. □

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